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1995 J. Phys. A: Math. Gen. 28 6717

(http://iopscience.iop.org/0305-4470/28/23/021)

J. Phys. A: Math. Gen. 28 (1995) 6717–6729. Printed in the UK

Integrable boundary conditions for the Toda lattice

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Received 9 May 1995

Abstract. The problem of construction of the boundary conditions for the Toda lattice compatible with its higher symmetries is considered. It is demonstrated that this problem is reduced to finding the differential constraints consistent with the ZS-AKNS hierarchy. A method of their construction is offered based on the Bäcklund transformations. It is shown that the generalized Toda lattices corresponding to the non-exceptional Lie algebras of finite growth can be obtained by imposing one of the four simplest integrable boundary conditions on both ends of the lattice. This fact allows, in particular, the solution of the reduction problem of the series A Toda lattices into the series D lattices. Deformations of the found boundary conditions are presented which lead to the Painlevé-type equations.

1. Introduction

In this paper we consider the boundary conditions for the classical completely integrable differential-difference model—the well known Toda lattice

$$q_{j,xx} = \exp(q_{j+1} - q_j) - \exp(q_j - q_{j-1}) \qquad -\infty < j < +\infty \tag{1}$$

which is compatible with the integrability property. Imposing boundary conditions of the form

$$q_M = F(q_{M+1}, \dots, q_{M+m}) \qquad q_N = G(q_{N-1}, \dots, q_{N-n}) \qquad M < N \tag{2}$$

reduces the infinite-dimensional system (1) to a finite-dimensional one. We require that the boundary conditions (2) are consistent with integrability at any choice of parameters $M < N$ (for fixed m, n). Thus, the boundary conditions on the left and right ends are assumed to be independent from each other; so actually we work with a semi-infinite lattice.

The compatibility of a boundary condition with the integrability property is understood, in the spirit of the symmetry approach, as being consistent with the flows determined by higher symmetries of the Toda lattice. (The exact definition is given below, in section 2.) Such interpretation was suggested in [1, 2] and has quite justified itself in the problem of description of the integrable boundary conditions for such models, as Harry Dym and Korteweg–de Vries (KdV) equations, multifield systems of the NLS and Burgers type, Volterra lattices and others (see [3–5]).

In some cases (for example, for the Burgers equation see [4]) one can prove that a boundary condition compatible at least with one higher symmetry of the equation is compatible with an infinite series of its symmetries as well, including a certain one, which

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0305-4470/95/236717+13$19.50

can easily be specified *a priori* and consequently may be regarded as a test symmetry. Compatibility with the test symmetry is taken as a basis of the preliminary classification of the integrable boundary conditions (see section 3).

The finite-dimensional versions of lattice (1), called generalized Toda lattices, are well known in the literature. Namely, the generalized Toda lattice, which is integrable in some sense, corresponds to each simple Lie algebra of the finite growth (see e.g. [6, 7]). In this paper we undertake an attempt to revise the theory of the finite-dimensional integrable systems of exponential type from the integrable boundary conditions point of view. For example, we demonstrate that the generalized Toda lattices corresponding to the classical series of finite-dimensional simple Lie algebras, as well as Kac–Moody algebras, can be represented as reductions of the infinite lattice (1) with the boundary conditions (2) compatible with its higher symmetries. It should be emphasized that the periodic lattices corresponding to the series $\tilde{A}$, as well as the lattices related to the exceptional Lie algebras, remain outside our consideration by virtue of the assumption about independence of the boundary conditions on the left and right ends. On the other hand, the approach developed in this work makes it possible to also study boundary conditions leading beyond the limits of exponential-type systems (see, for example, boundary condition (29) with a complete set of parameters). The revealing of the algebraic structures appropriate to such boundary conditions is an open problem.

The article is organized as follows. In section 2 the compatibility criterion of a boundary condition with a higher lattice symmetry is formulated. It involves the dual language of evolutionary partial differential equations associated with the lattice. In this section we also discuss the phenomenon of a degenerate boundary condition, which is characterized by increased integrability.

In section 3 we present a method to construct the integrable boundary conditions for the Toda lattice which is based on this criterion. Symmetries of the Toda lattice, rewritten in the evolutionary form, coincide with the ZS-AKNS hierarchy and the finding of integrable boundary conditions is equivalent to the finding of differential constraints compatible with the odd members of this hierarchy, in particular, with the system (11) which plays the role of the test symmetry. This problem is solved with the help of the Bäcklund transformations. It is noticed that, by virtue of the small order differential constraints found, system (11) is reduced into the integrable scalar equations KdV, mKdV and Calogero–Degasperis equations.

In section 4 we demonstrate that imposing every possible combination of the four simplest boundary conditions found above on the left and right ends of the Toda lattice leads to the generalized Toda lattices, corresponding to all infinite series of the finite growth Lie algebras (except for the $\tilde{A}$ series). Moreover, it is proved that the series D Toda lattices can be represented as reductions of the series A lattices. We shall note that although for the lattices of the series B and C the connection with the series A is quite obvious and well known, for the lattices of the series D this problem remains open†. Proposition 4.1 is proven: let the solution of the Toda lattice corresponding to the Lie algebra A_{2n-1} satisfy an additional symmetry of the reflection type, then it also sets the solution of the Toda lattice corresponding to the Lie algebra D_n.

In section 5 the found boundary conditions are deformed by the transform $q_j \to q_j + \varepsilon x$. The new boundary conditions are also integrable, but they are compatible with other symmetry sets. We demonstrate that closing lattice (1) by the boundary conditions with different values of parameter ε on the right and left ends results in equations of the Painlevé type.

† The authors thank R I Yamilov who drew their attention to this problem.

2. Boundary conditions compatible with higher symmetries

Let a lattice of the Toda type

$$q_{j,xx} = f(q_{j-1}, q_j, q_{j+1}) \qquad -\infty < j < +\infty \tag{3}$$

where $\frac{\partial f}{\partial q_{j\pm 1}} \neq 0$ admits a higher symmetry of the form

$$q_{j,t} = g(q_{j-k}, q_{j-k,x}, \dots, q_{j+k}, q_{j+k,x}). \tag{4}$$

Using the boundary condition

$$q_0 = F(q_1, q_{1,x}, q_2, q_{2,x}, \dots, q_m, q_{m,x}) \tag{5}$$

we shall reduce lattice (3) to a semi-infinite lattice determined on the right half-axis $j > 0$. Everywhere below we assume local analyticity of the functions f, g, F. Notice that, generally speaking, for $k > 1$ the boundary condition (5) is not sufficient to close lattice (4). For this purpose in addition to (5) it is necessary to express the variables $q_{-1}, q_{-1,x}, q_{-2}, q_{-2,x}, \dots, q_{1-k}, q_{1-k,x}$ through the dynamic variables $q_1, q_{1,x}, q_2, q_{2,x}, \dots$. Assuming that variables $q_0, q_{-1}, \dots, q_{1-k}$ also satisfy equation (3), one can easily obtain an algorithm of the boundary condition (5) continuation. First of all, differentiating (5), one finds an expression for $q_{0,x}$. Then, solving the equation $f(q_{-1}, q_0, q_1) = D_x^2(F)$ with respect to q_{-1}, one obtains

$$q_{-1} = F^{(-1)}(q_1, q_{1,x}, \dots, q_{m+1}, q_{m+1,x}).$$

Further, on each following step one finds $q_{-j} = F^{(-j)}(q_1, q_{1,x}, \dots)$ from equation

$$f(q_{-j}, F^{(1-j)}, F^{(2-j)}) = D_x^2(F^{(1-j)}).$$

Definition. The boundary condition (5) is called degenerate if the following identity holds:

$$\left.\frac{\partial f(q_{-1}, q_0, q_1)}{\partial q_{-1}}\right|_{q_0=F} \equiv 0.$$

Obviously, for the degenerate boundary condition the continuation algorithm is not valid.

Proposition 2.1. Let the boundary condition (5) be degenerate, then F does not depend on variables $q_{1,x}, q_2, q_{2,x}, q_3, q_{3,x}, \dots$ (but can depend on q_1).

Proof. Assume that variables q_1 and F are independent, i.e. F depends on q_m or $q_{m-1,x}$ $m > 1$. Then identity $\frac{\partial f}{\partial q_{-1}}(q_{-1}, F, q_1) \equiv 0$ yields that the function $f(q_{j-1}, q_j, q_{j+1})$ does not depend on variable q_{j-1} which contradicts the form of the lattice (3). □

Consequence. Let the boundary condition (5) be not degenerate, then the algorithm of continuation given above is correct.

Proof. In the identity $\frac{\partial f}{\partial q_{-j}}(q_{-j}, F^{(1-j)}, F^{(2-j)}) \equiv 0$ the functions $F^{(2-j)}(q_1, \dots, q_{m+j-2})$ and $F^{(1-j)}(q_1, \dots, q_{m+j-1})$ depend on the different sets of dynamic variables and therefore are independent. Hence, from this identity the contradiction follows that the function f in (3) essentially depends on the third argument. □

Proposition 2.2. Let the boundary condition (5) be degenerate, then it completely closes the lattice (4) on the half-axis $j > 0$.

Proof. From the degeneracy of the boundary condition (5) it follows that variables $q_{-1}, q_{-1,x}$, $q_{-2}, q_{-2,x}, \dots$ cannot be expressed through the dynamic variables $q_1, q_{1,x}, q_2, q_{2,x}, \dots$ of the problem (3), (5). Therefore they can be considered as independent variables. From the condition of commutation of the flows determined by equations (3), (4) in the point $j = 1$, one obtains $D_t(f_1) = D_x^2(g_1)$ where the subscript denotes shift on j:

$$f_j = f(q_{j-1}, q_j, q_{j+1}) \qquad g_j = g(q_{j-k}, q_{j-k.x}, \ldots, q_{j+k}, q_{j+k.x}) .$$

Expanding this relation one obtains

$$\frac{\partial f_1}{\partial q_0} D_t(F) + \frac{\partial f_1}{\partial q_1} g_1 + \frac{\partial f_1}{\partial q_2} g_2 = \frac{\partial g_1}{\partial q_{1-k}} f_{1-k} + \frac{\partial g_1}{\partial q_{1-k.x}} D_x(f_{1-k}) + \cdots + \frac{\partial g_1}{\partial q_{1+k}} f_{1+k} + \frac{\partial g_1}{\partial q_{1+k.x}} D_x(f_{1+k}) + R$$

where R contains all terms with the second-order partial derivatives of g_1. It is easy to see that the left-hand side of the last equality does not depend on the variables $q_{-k}, q_{-k,x}$, therefore the term $\frac{\partial g_1}{\partial q_{1-k.x}} \frac{\partial f_{1-k}}{\partial q_{-k}} q_{-k.x}$ on the right-hand side is equal to 0, under the condition (5). But for $k > 1$ it is obvious that $\frac{\partial f_{1-k}}{\partial q_{-k}}|_{(5)} \neq 0$, therefore $\frac{\partial g_1}{\partial q_{1-k.x}}|_{(5)} = 0$. The factor $\frac{\partial g_1}{\partial q_{1-k}}$ is also equal to 0, for otherwise the first term on the right-hand side of the equality would essentially depend on the variable q_{-k}.

Thus, under condition (5) the function g_1 does not depend on $q_{1-k}, q_{1-k.x}$. Assembling factors at independent variables $q_{1-k}, q_{1-k.x}, \ldots, q_{-2}, q_{-2.x}$ and repeating the reasoning, one gets that $\frac{\partial g_1}{\partial q_{-j}}|_{(5)} \equiv 0$, $\frac{\partial g_1}{\partial q_{-j.x}}|_{(5)} \equiv 0$, $j = 1, 2, \ldots, k-1$, i.e. the right-hand side of the equation $q_{1.t} = g(q_{1-k}, q_{1-k.x}, \ldots, q_{1+k}, q_{1+k.x})$ under condition (5) actually depends only on dynamic variables $q_1, q_{1.x}, q_2, q_{2.x}, \ldots$. The fact that other equations of the lattice (4) for $n > 0$ do not depend on variables $q_{-j}, q_{-j.x}$ if $q_0 = F$ is checked similarly. □

Consequence. The degenerate boundary condition reduces commuting infinite lattices (3), (4) into commuting semi-infinite lattices, given on the half-axis $j > 0$.

Notice that a similar fact is true for the lattices $q_{j,x} = f(q_{j+1}, q_j, q_{j-1})$ of the Volterra type as well.

Definition. The boundary problem (3), (5) is called compatible with the higher symmetry (4), if one of the following conditions is fulfilled:

(i) boundary condition (5) is degenerate;
(ii) semi-infinite lattice reduced from (4) by virtue of (5) and its differential consequences obtained by differentiating it with respect to x by virtue of (3) (see above algorithm of continuation) commutes with the semi-infinite lattice (3), (5).

One can associate with the pair of lattices (3), (4) a system of two partial differential equations. For this purpose one passes from the standard set of dynamical variables $q_0, q_{0.x}, q_{\pm 1}, q_{\pm 1.x}, \ldots$ to the dynamical set consisting of the variables q_0, q_1 and their derivatives with respect to x. Thus q_{-1} and q_2 are expressed from the conditions $q_{0.xx} = f(q_1, q_0, q_{-1})$ and $q_{1.xx} = f(q_2, q_1, q_0)$ and so on. Rewriting the lattice (4) in the new variables, one comes to a system of evolutionary equations (cf [8])

$$\begin{aligned} q_{0.t} &= g_+(q_0, q_1, q_{0.x}, q_{1.x}, \ldots, q_{0.x\ldots x}, q_{1.x\ldots x}) \\ q_{1.t} &= g_-(q_0, q_1, q_{0.x}, q_{1.x}, \ldots, q_{0.x\ldots x}, q_{1.x\ldots x}) . \end{aligned} \tag{6}$$

This transformation maps the boundary condition (5) into the differential constraint of the form

$$q_0 = \hat{F}(q_1, q_{0.x}, q_{1.x}, \ldots, q_{0.x\ldots x}, q_{1.x\ldots x}) . \tag{7}$$

The following criterion of the boundary condition compatibility with the higher symmetry is a direct consequence of the way the system (6) was constructed.

Proposition 2.3. In order that the boundary problem (3), (5) be compatible with the symmetry (4), it is necessary and sufficient that the differential constraint (7) be consistent with the dynamics of the equation (6).

3. Differential constraints compatible with the ZS-AKNS hierarchy

Let us rewrite the higher symmetries of the Toda lattice

$$q_{j,t_2} = q_{j,x}^2 + \exp(q_{j+1} - q_j) + \exp(q_j - q_{j-1}) \tag{8}$$

$$q_{j,t_3} = q_{j,x}^3 + (2q_{j,x} + q_{j+1,x}) \exp(q_{j+1} - q_j) + (2q_{j,x} + q_{j-1,x}) \exp(q_j - q_{j-1}) \tag{9}$$

as the systems of partial differential equations, introducing new variables $u_j = \exp q_{j+1}$, $v_j = \exp(-q_j)$ and expressing $q_k, q_{k,x}$ through derivatives of these variables with respect to x owing to the Toda lattice (see [8]). By virtue of the lattice (8) each pair of the variables u_j, v_j satisfies the ZS-AKNS system

$$u_{t_2} = u_{xx} + 2u^2 v \qquad -v_{t_2} = v_{xx} + 2v^2 u \tag{10}$$

and by virtue of the lattice (9)—the system of coupled KdV equations

$$u_{t_3} = u_{xxx} + 6uvu_x \qquad v_{t_3} = v_{xxx} + 6vuv_x. \tag{11}$$

Other higher symmetries of the Toda lattice also correspond to the higher symmetries from ZS-AKNS hierarchy. According to proposition 2.3, the finding of a boundary condition of the form (5) compatible with some higher symmetry of the Toda lattice is reduced to the finding of a differential constraint of the form (7) compatible with dynamics, determined by this symmetry. Rewriting this constraint in variables u_0, v_0 and omitting the subscript 0 for brevity, we come to the following problem.

Problem. Find the differential constraints of the form

$$F(u, v, u_x, v_x, \ldots, u_{x\ldots x}, v_{x\ldots x}) = 0 \tag{12}$$

compatible with the kth symmetry of the ZS-AKNS hierarchy, that is those satisfying the identity

$$D_{t_k}(F)|_{F=0} \equiv 0. \tag{13}$$

Let us call the order of the differential constraint (12) the maximum order of derivatives with respect to x contained in it. Finding the constraint of the given order compatible with a given flow is reduced to direct, but rather tedious, calculations. However, it is easy to prove that because of skew symmetry of the main part, the system (10) and its even order symmetries do not admit any differential constraint, except trivial ones

$$u = 0 \qquad \text{or} \qquad v = 0 \tag{14}$$

which correspond to the boundary conditions $\exp(q_1) = 0$ or $\exp(-q_0) = 0$. Thus, the problem can have a non-trivial solution only for the flows appropriate to the times t_{2k+1}. Apparently, if the constraint (12) is compatible with some odd-order higher symmetry, then it is compatible with the system (11) as well. Based on this hypothesis, we shall use the system (11) as a test symmetry when constructing boundary conditions for the Toda lattice. The following statements are proven by direct computations.

Proposition 3.1. The constraint $u = P(v)$ is compatible with dynamics of the equation (11) only the the case when P is a linear function:

$$u = \alpha v + \beta. \tag{15}$$

Proposition 3.2. The differential constraint of the first order is compatible with the system (11) if and only if it is of the form $u_x = a(u, v)v_x$, where the factor $a(u, v)$ satisfies the Hopf equation $a_v + aa_u = 0$.

Remark. In essence, this constraint is reduced to the linear constraint from the previous proposition. Really, the function a is the first integral of relation $u_x = av_x$; assuming $a = \alpha$ and integrating, one obtains (15).

Proposition 3.3. The differential constraint of the form $u = P(v, v_x, v_{xx})$ compatible with the system (11) is set by the formula

$$u(v^2 - c_0) = \frac{v_x^2 v + c_2(v^2 + c_0) + c_1 v}{v^2 - c_0} - v_{xx} \tag{16}$$

where c_0, c_1, c_2 are arbitrary constants.

Finding more complex constraints directly along the definition (13) becomes difficult. For further progress we need a method which would allow us to reproduce new examples from already found ones. For this purpose it is enough to find operations acting on the set of integrable constraints. Two such operations are obvious. Indeed, the system (11) is invariant under the transformations

$$S_c:\ \tilde{u} = cu \qquad \tilde{v} = v/c \qquad c = \text{constant} \tag{17}$$

$$R:\ \tilde{u} = v \qquad \tilde{v} = u\,. \tag{18}$$

Hence rewriting the constraint (12) in the new variables yields the integrable constraint again. For example, in (16) it is possible to swap the positions of u and v. The less obvious operation is application of the Bäcklund (auto-)transformations. As is well known (see, e.g. [8]) the system (11) admits two essentially different Bäcklund transformations. One of them is given by the explicit formula

$$T:\ v_1 = 1/u \qquad u_1 = u_{xx} - u_x^2/u + u^2 v \tag{19}$$

and is equivalent to the shift $q_j \to q_{j+1}$ in the Toda lattice. Substituting expression (19) into the constraint (12) imposed on the variables u_1, v_1 one finds a new constraint on the variables u, v. In this case the orders of both constraints differ not more than by two. For example, from the constraint $u_1 = 1$ one obtains (using in addition the reflection R) the constraint (16) with constants $c_0 = c_1 = 0$, $c_2 = 1$.

The second Bäcklund transformation (more precisely, its 'x-part') is of the form

$$B_\mu:\ u_x = \tilde{u} + \mu u + u^2\tilde{v} \qquad -\tilde{v}_x = v + \mu\tilde{v} + \tilde{v}^2 u\,. \tag{20}$$

Differentiation of these relations gives expressions for v and derivatives of u and v of any order through u and derivatives of $\tilde{u}, \tilde{v}$. Substituting these expressions into the constraint (12), one obtains some relation of the form $\hat{F}(u, \tilde{u}, \tilde{v}, \ldots, \tilde{u}_{x\ldots x}, \tilde{v}_{x\ldots x}) = 0$. Eliminating the variable u from the equations $\hat{F} = 0$ and $D_x(\hat{F}) = 0$ one obtains a differential constraint for the variables $\tilde{u}, \tilde{v}$

$$\tilde{F}(\tilde{u}, \tilde{v}, \ldots, \tilde{u}_{x\ldots x}, \tilde{v}_{x\ldots x}) = 0\,.$$

According to the definition of the Bäcklund transformation, the new variables $\tilde{u}, \tilde{v}$ also satisfy the system (11), hence the found constraint appears compatible with this system as well. We call the described procedure the dressing of the constraint (12). It is easy to check that dressing also changes the order of constraint by no more than two.

The indicated transformations generate some group with relations

$$\begin{aligned} &S_c S_d = S_{cd} \qquad RS_cR = S_{\frac{1}{c}} \qquad S_cT = TS_c \qquad S_cB_\mu = B_\mu S_c \qquad R^2 = 1 \\ &RTR = T^{-1} \qquad RB_\mu R = S_{-1}B_{-\mu}^{-1} \qquad TB_\mu = B_\mu T \qquad B_\mu B_\nu = B_\nu B_\mu\,. \end{aligned} \tag{21}$$

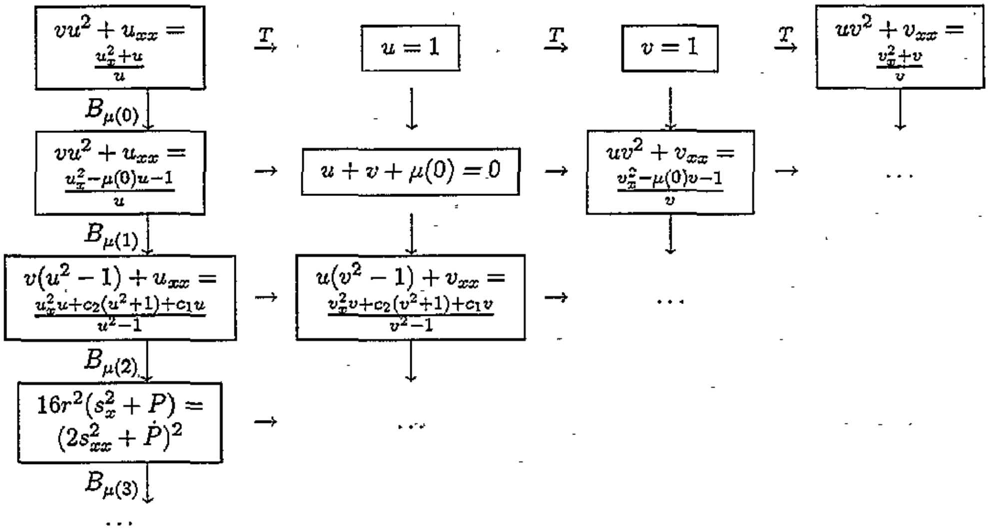

Figure 1. Dressing of the differential constraints.

Our hypothesis is that all constraints compatible with the odd flows of the ZS-AKNS hierarchy form the orbit of two simplest constraints $u = 1$ and $u = 0$ under action of this group. Notice that for all constraints obtained in such a way the consistence with the odd flows really takes place. It follows from the facts that these flows are compatible with the constraint $u =$ constant and that the transformations (17)–(20) act on all hierarchy. The constraints obtained as a result of application of several transformations T and B_μ to the non-degenerate constraint $u = 1$ are listed in the figure 1. Let us comment on the formulae indicated in it. Denoting the result of the mth iteration of the Bäcklund transformation through $u(m)$, $v(m)$ one obtains from (20) a lattice

$$u_x(m) = u(m+1) + \mu(m)u(m) + u^2(m)v(m+1) \tag{22}$$

$$-v_x(m+1) = v(m) + \mu(m)v(m+1) + v^2(m+1)u(m)\,. \tag{23}$$

It is obvious that by virtue of the constraint $u(0) = 1$ the equation (22) at $m = 0$ turns into the constraint $u(1) + v(1) + \mu(0) = 0$ of the form (15). Notice that variables $v(0)$, $v(1)$ satisfy the equations KdV and mKdV

$$v_t(0) = v_{xxx}(0) + 6v(0)v_x(0) \qquad v_t(1) = v_{xxx}(1) - 6(v(1) + \mu(0))v(1)v_x(1)$$

respectively, and the equation (23) at $m = 0$ turns into the Miura map

$$-v(0) = v_x(1) + \mu(0)v(1) + v^2(1)$$

between them. On the next step the equation (23) at $m = 1$ yields

$$u(1) = -\frac{v_x(2) + \mu(1)v(2) - \mu(0)}{v^2(2) - 1} \tag{24}$$

and, substituting in (22) at $m = 1$ one obtains, after simple calculations, the constraint (16) with constants

$$c_0 = 1 \qquad c_1 = -\mu^2(0) - \mu^2(1) \qquad c_2 = \mu(0)\mu(1)\,.$$

written for the variables $u(2)$, $v(2)$. Notice that using the transformation (17) one can obtain constraint (16) with an arbitrary choice of parameters. Variable $v(2)$ satisfies the equation

$$v_t = v_{xxx} - \frac{6vv_x}{v^2-1}\left(v_{xx} - \frac{vv_x^2 + c_2(v^2+1) + c_1 v}{v^2-1}\right). \tag{25}$$

Equation (24) can be rewritten in the form of differential substitution

$$v(1) = \frac{v_x(2) + \mu(1)v(2) - \mu(0)v^2(2)}{v^2(2)-1}$$

connecting mKdV and equation (25). Notice, that the point transformation $v = \tanh y$ brings (25) into the so called exponential Calogero–Degasperis equation [9]

$$y_t = y_{xxx} - 2y_x^3 - \tfrac{3}{8}\big((\mu(1)-\mu(0))^2 e^{4y} + (\mu(1)+\mu(0))^2 e^{-4y} - 2\mu^2(1) - 2\mu^2(0)\big)y_x .$$

Further application of the transformation (20) results into a cumbersome constraint of fourth order. It appears, however, that the combination $T^{-1}B_{\mu(2)}$ gives constraint of the second order again. It can conveniently be written in the variables

$$2r = u_{-1}(3) - v_{-1}(3) \qquad 2s = u_{-1}(3) + v_{-1}(3) .$$

Omitting rather bulky calculations we give only the answer

$$16r^2(s_x^2 + P) = (2s_{xx} + \dot{P})^2 \tag{26}$$

where

$$P = (s-\alpha)(s-\alpha_0)(s-\alpha_1)(s-\alpha_2) \qquad 2\alpha = \mu(0) + \mu(1) + \mu(2) \qquad \alpha_j = \mu(j) - \alpha .$$

Rewriting the system (11) in the new variables and eliminating r by virtue of the constraint one finds that variable s satisfies the elliptic Calogero–Degasperis equation

$$s_t = s_{xxx} + 6\left(s^2 - \frac{(2s_{xx} + \dot{P})^2}{16(s_x^2 + P)}\right)s_x$$

also introduced in [9]. The differential substitution connecting this equation with equation (25) is given by the formula

$$(c_2 + 2\mu(2)s - 2s_x)v^2 + (c_1 + \mu(2)^2 + 4s^2)v + c_2 + 2\mu(2)s + 2s_x = 0 .$$

Differential substitutions for the scalar evolution equations are described in detail in the literature (see e.g. [10]).

4. Boundary conditions for the Toda lattice

Let us write down the boundary conditions corresponding to the differential constraints (14)–(16) found above (certainly the constraint (26) gives some boundary condition as well, but we do not need it). Passing to the variables q_j one obtains

$$\exp(q_1) = 0 \quad \text{or} \quad \exp(-q_0) = 0 \tag{27}$$

$$\exp(q_1) = \alpha\exp(-q_0) + \beta \tag{28}$$

$$c_0\exp(q_1) = \exp(-q_{-1}) + \frac{c_0 q_{0,x}^2 + c_2(c_0\exp(q_0) + \exp(-q_0)) + c_1}{c_0\exp(q_0) - \exp(-q_0)} \tag{29}$$

where $\alpha, \beta, c_0, c_1, c_2$ are arbitrary parameters. It should be mentioned that the boundary conditions (27), (28) were found earlier in [11] and [12], respectively.

It is well known (see, e.g. [6, 7, 13]) that each simple Lie algebra of the finite growth corresponds to some integrable generalized Toda lattice. The lattices associated with the

Lie algebras A_{n-1}, B_n, C_n are obtained by imposing the degenerate boundary condition $\exp(-q_0) = 0$ on the left end of the lattice (1) and boundary conditions (27), (28) of particular form on its right end, namely $\exp(q_{n+1}) = 0$, $q_{n+1} = 0$, $q_{n+1} = -q_n$, respectively. Now we shall demonstrate that lattices of the D type correspond to the boundary condition of the form (29). The Lie algebra D_n corresponds to the lattice

$$\begin{aligned} q_{1,xx} &= e^{q_2-q_1}, \\ q_{j,xx} &= e^{q_{j+1}-q_j} - e^{q_j-q_{j-1}} \qquad j=2,\ldots,n-2 \\ q_{n-1,xx} &= (e^{q_n}+e^{-q_n})e^{-q_{n-1}} - e^{q_{n-1}-q_{n-2}} \\ q_{n,xx} &= (e^{-q_n}-e^{q_n})e^{-q_{n-1}}. \end{aligned} \tag{30}$$

It is clear that the boundary condition on the left end is of the form $e^{-q_0} = 0$, i.e. is degenerate. In order to bring the last but one equation of system (30) into the standard form we make the change of variables

$$\hat{q}_j = q_j - \log 2 \qquad j = 1,\ldots,n-1 \qquad e^{\hat{q}_n} = \cosh q_n. \tag{31}$$

Thus the last equation of the system accepts the form (we omit the hat over variables q_j)

$$q_{n,xx} = (e^{-q_n} - e^{q_n})e^{-q_{n-1}} + \frac{q_{n,x}^2}{e^{2q_n}-1}.$$

Assuming $q_{n,xx} = e^{q_{n+1}-q_n} - e^{q_n-q_{n-1}}$ one obtains a boundary condition for the right end of the lattice:

$$e^{q_{n+1}} = e^{-q_{n-1}} + \frac{q_{n,x}^2}{2\sinh q_n}. \tag{32}$$

Obviously, up to the shift $q_0 \mapsto q_n$, the formula (32) is a particular case of (29) for $c_0 = 1$, $c_1 = c_2 = 0$. So, the Toda lattice of the type D_n is reduced, by use of the change of variables (31), to the following finite-dimensional reduction of the Toda lattice (1):

$$\begin{aligned} e^{-q_0} &= 0 \\ q_{j,xx} &= e^{q_{j+1}-q_j} - e^{q_j-q_{j-1}} \qquad j=1,\ldots,n \\ e^{q_{n+1}} &= e^{-q_{n-1}} + \frac{q_{n,x}^2}{2\sinh q_n}. \end{aligned} \tag{33}$$

Now we can represent systems related to the Lie algebras $A_{n-1}, B_n, C_n, D_n, BD_n, BA_{2n-2}$, $\tilde{B}_{n-1}, \tilde{C}_{n-1}, CA_{2n-3}, \tilde{D}_{n-1}$ (in other words, only the series $\tilde{A}$ corresponding to the periodic Toda lattice and exceptional Lie algebras drop out of our consideration) in the form of the problem (1), (2) with boundary conditions of the type (27)–(29). Actually in this matter only the structure of the beginning and the end of the Lie algebra Coxeter–Dynkin diagram is important. The left (right) column of figure 2 contains the boundary conditions on the left (right) end of the lattice and the corresponding variants of the beginning (end) of the diagram. All possible combinations of the boundary conditions on the left and right ends give us the Toda lattices appropriate to all Lie algebras listed above.

We shall demonstrate now that the series D lattices can be obtained from the series A lattices as a result of reflection-type reduction combined with the Bäcklund transformation (20). For this purpose we need to rewrite the transformations (17)–(20) in the variables q_j:

$$\begin{aligned} &S_c: \ \tilde{q}_j = q_j + \log c \qquad R: \ \tilde{q}_j = -q_{1-j} \qquad T: \ \tilde{q}_j = q_{j+1} \\ &B_\mu: \ \begin{cases} \tilde{q}_{j,x} = \exp(\tilde{q}_j - q_j) + \exp(q_{j+1} - \tilde{q}_j) + \mu \\ q_{j,x} = \exp(\tilde{q}_j - q_j) + \exp(q_j - \tilde{q}_{j-1}) + \mu. \end{cases} \end{aligned} \tag{34}$$

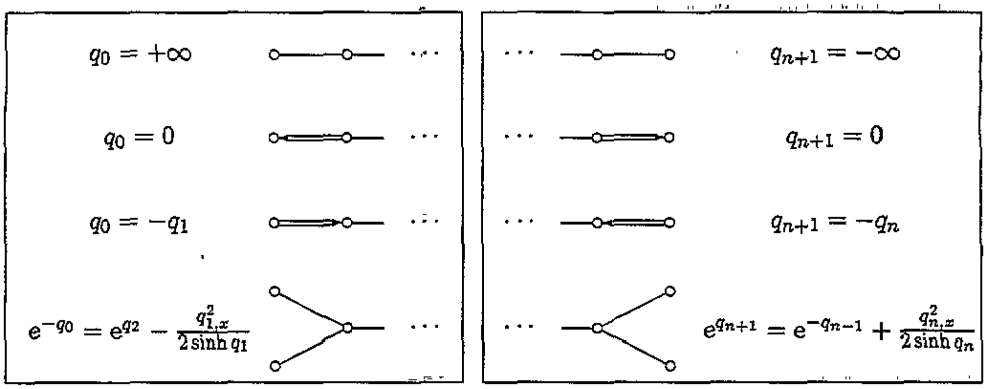

Figure 2. Boundary conditions corresponding to the ends of Coxeter–Dynkin diagrams.

The generalized Toda lattice corresponding to the Lie algebra A_{m-1} is obtained from the infinite lattice by imposing two degenerate boundary conditions:

$$\begin{aligned} &q_0 = +\infty \\ &q_{j,xx} = e^{q_{j+1}-q_j} - e^{q_j-q_{j-1}} \qquad j = 1, \ldots, m \\ &q_{m+1} = -\infty . \end{aligned} \tag{35}$$

It is easy to check that the formula (34) complemented by the conditions

$$\tilde{q}_0 = q_0 = +\infty \qquad \tilde{q}_{m+1} = q_{m+1} = -\infty \tag{36}$$

sets the Bäcklund transformation for the system (35). Thus the mapping $B_\mu : Q \mapsto \tilde{Q}$, where $Q = (q_1, q_{1,x}, \ldots, q_m, q_{m,x})$ is given by explicit formulae and is convertible. Indeed, the relations (34), (36) represent a closed system of algebraic equations relatively $\tilde{q}_{j,x}, \exp(\tilde{q}_j)$, which can easily be solved recurrently, starting from the equation $q_{1,x} = \exp(\tilde{q}_1 - q_1) + \mu$. Notice that the given transformation is an example of integrable discrete mapping. It can be interpeted as a finite-dimensional reduction of the twice discrete lattice induced by transformation (34). For details concerning integrable mappings see, for example, [14–16].

The following statement establishes the connection between lattices of the series A and D.

Proposition 4.1. Let $m = 2n$ and the solution Q of the system (35) satisfy the reduction condition $Q = S_{-1}RT^m Q$, that is $q_j = i\pi - q_{m+1-j}$. Then the new solution $\tilde{Q} = B_0 Q$ of the system (35) possesses the following properties:

(1) $\tilde{Q} = B_0^2 RT^m \tilde{Q}$,;

(2) the first half of the vector $\tilde{Q}$, that is the vector $(\tilde{q}_1, \tilde{q}_{1,x}, \ldots, \tilde{q}_n, \tilde{q}_{n,x})$, solves the lattice (33).

Proof. Property (1) can easily be proven with the help of relations (21). Further, from reduction conditions on q it follows that $q_{n+1} = i\pi - q_n$, that is $u_n = -v_n$. It follows from figure 1 that dressing of this constraint by transformation B_0 yields the constraint of the form (16) with constants $c_0 = 1$, $c_1 = c_2 = 0$ connecting the variables $\tilde{u}_n, \tilde{v}_n$. As we have already seen this constraint is equivalent to the boundary condition (32). □

Concluding this section we note that the similar simple connection can be established between the type A lattices and the lattices with the boundary condition (29) of general form on the right end. The proof of the following proposal does not cause difficulties.

Proposition 4.2. Let $m = 2n+1$ and the solution Q of the system (35) satisfy the reduction condition $q_j = -q_{m+1-j}$. Then the new solutions of this system $\tilde{Q} = S_c B_\mu Q$, $\bar{Q} = B_\nu \tilde{Q}$ where c, μ, ν are arbitrary parameters solve the Toda lattices with the boundary condition $\exp(-q_0) = 0$ on the left end and boundary conditions (28), (29) of the general form on the right end, respectively.

It follows from the well known fact on the general solution of the type A_m lattice (see, e.g. [6]) and propositions 4.1 and 4.2 that the general solutions of such lattices can be written explicitly as rational functions on the exponents $e^{\lambda x}$.

5. Boundary conditions with explicit dependence on x

The boundary conditions found above admit some generalization. Notice that the Toda lattice (1) is invariant under transformation

$$X: \ \tilde{q}_j = q_j + \varepsilon x \tag{37}$$

or, in the variables u, v

$$X: \ \tilde{u} = e^{\varepsilon x} u \qquad \tilde{v} = e^{-\varepsilon x} v . \tag{38}$$

The difference between this transformation and transformations S_c, R, T, B_μ considered earlier is that it does not preserve the higher symmetries of the Toda lattice. For example, the system (11) at this change of variables is rewritten as follows:

$$u_{t_3} = u_{xxx} + 6uvu_x - 3\varepsilon(u_{xx} + 2u^2 v) + 3\varepsilon^2 u_x - \varepsilon^3 u$$
$$v_{t_3} = v_{xxx} + 6uvv_x + 3\varepsilon(v_{xx} + 2v^2 u) + 3\varepsilon^2 v_x + \varepsilon^3 v .$$

In general, it is easy to check that change (38) determines some linear transformation in the space of symmetries of the ZS-AKNS hierarchy. Boundary conditions found in the previous section are consistent with the odd-order symmetries subspace. If this subspace were *invariant under the transformation X then rewriting these boundary conditions in variables* $\tilde{q}_j$ would give nothing new. The presence of the even-order terms in the transformed symmetries results in that the transformed boundary conditions contain explicit dependence on x. For example, the formulae (28), (29) are converted into the formulae

$$\exp(q_1) = \alpha \exp(2\varepsilon x - q_0) + \beta \exp(\varepsilon x) \tag{39}$$

$$c_0 \exp(q_1) = \exp(2\varepsilon x - q_{-1}) + \frac{c_0(q_{0,x} - \varepsilon)^2 + c_2(c_0 \exp(q_0 - \varepsilon x) + \exp(\varepsilon x - q_0)) + c_1}{c_0 \exp(q_0 - 2\varepsilon x) - \exp(-q_0)} \tag{40}$$

correspondingly. It should be emphasized that new boundary conditions are not worse than old ones, but they are compatible with other symmetry sets. On imposing on both ends of the Toda lattice of boundary conditions relevant to the same value ε, this symmetry set passes into the symmetries of obtained finite-dimensional systems and provides its integrability. The situation, however, essentially changes if the boundary conditions imposed on the different ends of the lattice correspond to the different values of parameter ε. It turns out that in small dimensions the equations of the Painlevé type arise (compare with [18], where the Painlevé equations arise at quasiperiodic closing of integrable lattices). In the examples below the length-1 reductions of the Toda lattice are considered. On the left end, without loss of generality, we impose boundary conditions of the type (28) or (29) and on the right one deformed conditions (39) or (40) at some fixed value of ε. (It is clear that the boundary condition (27) will not give anything new, since the transformation (37) does not change it.)

(1) Let us try boundary conditions of the form (28) and (39) at $\varepsilon = 2$:

$$e^{-q_{-1}} = \alpha e^{q_0} + \beta \qquad e^{q_1} = \gamma e^{4x-q_0} + \delta e^{2x}.$$

On the variable $q = q_0$ one obtains the differential equation

$$q_{xx} = \gamma e^{4x-2q} + \delta e^{2x-q} - \alpha e^{2q} - \beta e^{q}.$$

It is easy to check that the point transformation $e^{q(x)} = zy(z)$, $e^x = z$ brings it into the third Painlevé equation

$$y_{zz} = \frac{y_z^2}{y} - \frac{y_z}{z} + \frac{1}{z}(Ay^2 + B) + Cy^3 + \frac{D}{y}$$

with the values of parameters $A = -\beta$, $B = \delta$, $C = -\alpha$, $D = \gamma$.

(2) Impose the boundary condition of the form (29) with $c_0 = 1$ without loss of generality on the left end and a boundary condition (39) with $\varepsilon = 1$ on the right one:

$$e^{-q_{-1}} = e^{q_1} - \frac{q_{0,x}^2 + c_2(e^{q_0} + e^{-q_0}) + c_1}{e^{q_0} - e^{-q_0}} \qquad e^{q_1} = \alpha e^{2x-q_0} + \beta e^x.$$

On the variable $q = q_0$ one obtains the differential equation

$$q_{xx} = \frac{q_x^2}{1 - e^{-2q}} + \alpha e^{2x}(e^{-2q} - 1) + \beta e^x(e^{-q} - e^{q}) + \frac{c_2(e^{q} + e^{-q}) + c_1}{1 - e^{-2q}}.$$

It is easy to check that the point transformation $e^{q(x)} = \frac{y(z)-1}{y(z)+1}$, $e^x = z$ brings it into the fifth Painlevé equation

$$y_{zz} = \left(\frac{1}{2y} + \frac{1}{y-1}\right)y_z^2 - \frac{y_z}{z} + \frac{(y-1)^2}{z^2}\left(Ay + \frac{B}{y}\right) + C\frac{y}{z} + D\frac{y(y+1)}{y-1}$$

with the values of parameters $8A = -c_1 - 2c_2$, $8B = c_1 - 2c_2$, $C = 2\beta$, $D = 2\alpha$.

(3) Finally, consider a combination of boundary conditions of the form (29) and (40) (thus, because of small dimension of the system, the left and right ends actually coincide). It is easy to prove that from seven parameters contained in these two formulae, three can be replaced by any non-zero constants by the use of dilating x, shift x and shift q. Therefore we may consider boundary conditions

$$e^{q_1} = e^{-q_{-1}} + \frac{q_{0,x}^2 + a(e^{q_0} + e^{-q_0}) + b}{e^{q_0} - e^{-q_0}}$$
$$e^{q_1} = e^{2x-q_{-1}} + \frac{(q_{0,x} - 1)^2 + c(e^{q_0-x} + e^{x-q_0}) + d}{e^{q_0-2x} - e^{-q_0}}.$$

Solving these equations relatively $e^{q_1}, e^{-q_{-1}}$ and substituting found values into the formula $q_{0,xx} = e^{q_1-q_0} - e^{q_0-q_{-1}}$, one obtains some second-order differential equation on the variable $q = q_0$. One can check that change of variables $e^{q(x)} = \frac{y(z)+\sqrt{z}}{y(z)-\sqrt{z}}$, $e^x = \frac{1+\sqrt{z}}{1-\sqrt{z}}$ converts it into the sixth Painlevé equation

$$y_{zz} = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-z}\right)y_z^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{y-z}\right)y_z + \frac{y(y-1)(y-z)}{z^2(z-1)^2}\left(A + B\frac{z}{y^2} + C\frac{z-1}{(y-1)^2} + D\frac{z(z-1)}{(y-z)^2}\right)$$

with parameters $8A = -b - 2a$, $8B = b - 2a$, $8C = -d - 2c$, $8D = d - 2c + 4$.

Acknowledgments

The authors thank A B Shabat, S I Svinolupov, V V Sokolov and R I Yamilov for their interest in this work and useful discussions. This work was partially supported by the Russian Foundation of Fundamental Researches (grants 93-011-16088 and 93-011-165) and the International Scientific Foundation (grants MLY000 and RK-2000).

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